Online Appendices for
"The Benchmark Inclusion Subsidy"

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Appendix B

In this appendix we explore the robustness of our model to an alternative specification where a manager's compensation is tied to the per-dollar returns on the fund and on the benchmark portfolio as opposed to the performance measure used in the main text.

Define $R_i = D_i/(\bar{x}_i S_i)$, $i = 1, \ldots, n$, and let $R = (R_1, \ldots, R_n)^\top$ be the vector of (per-dollar) returns. It is distributed normally with mean $\mu_R = (\mu_1/(\bar{x}_1 S_1), \ldots, \mu_n/(\bar{x}_n S_n))$ and variance Σ_R, where $(\Sigma_R)_{ij} = \rho_{ij}\sigma_i\sigma_j/(\bar{x}_i S_i \bar{x}_j S_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, n$.

It is now more convenient to specify investors' portfolio optimization problem in terms of fractions φ_i of wealth under management invested in stock i, $i = 1, \ldots, n$, with the remaining fraction $1 - \sum_{i=1}^n \varphi_i$ invested in the bond. Denote $\varphi = (\varphi_1, \ldots, \varphi_n)^\top$.

Let us start by considering the problem of a direct investor. Let W_0^D denote the initial wealth of each direct investor. Let $\mathbb{1} = (1, \ldots, 1)^\top$ be a vector of ones. As in main model, CARA preferences with normal returns are equivalent to mean-variance preferences. Then the direct investor's problem can be written as $\max_\varphi \left(\varphi^\top \mu_R + 1 - \mathbb{1}^\top \varphi\right) W_0^D - (\gamma/2)\varphi^\top \Sigma_R \varphi \left(W_0^D\right)^2$. The optimal solution is $\varphi^D W_0^D = \Sigma_R^{-1}\mu_R - \mathbb{1}/\gamma$.

Now consider fund managers. Suppose each manager is given W_0^M amount of money to manage, which is all or part of the fund investor's initial wealth. The manager's compensation is $w = [aR_\varphi + b(R_\varphi - R_{\mathbf{b}})]W_0^M + c$, where $R_\varphi = \varphi^\top R + 1 - \mathbb{1}^\top \varphi$ is the return on the manager's portfolio, and $R_{\mathbf{b}} = \omega^\top R$ is the benchmark return. The benchmark weights (defined as in Lemma 4 in Appendix A) are $\omega_i = \mathbf{1}_i \bar{x}_i S_i / \sum_{j=1}^n \mathbf{1}_j \bar{x}_j S_j$, and $\omega = (\omega_1, \ldots, \omega_n)^\top$. Then the manager's compensation can be written as $w = \left[(a+b)(\varphi^\top R + 1 - \mathbb{1}^\top \varphi) - b\omega^\top R\right] W_0^M + c$, and the manager's problem is

$$\max_\varphi \left[(a+b)(\varphi^\top \mu_R + 1 - \mathbb{1}^\top \varphi) - b\omega\mu_R\right] W_0^M - \frac{\gamma}{2}[(a+b)\varphi - b\omega]^\top \Sigma_R [(a+b)\varphi - b\omega] \left(W_0^M\right)^2.$$

The optimal solution is $\left[(a+b)\varphi^M - b\omega\right] W_0^M = \Sigma_R^{-1}(\mu_R - \mathbb{1})/\gamma$. Equating total demand with total supply, $\lambda_M \varphi^M W_0^M + \lambda_D \varphi^D W_0^D = \bar{x} \cdot S$, and rearranging terms, we arrive at the following representation of the stocks' expected returns:

$$\begin{pmatrix} \frac{\mu_1}{\bar{x}_1 S_1} - 1 \\ \vdots \\ \frac{\mu_n}{\bar{x}_n S_n} - 1 \end{pmatrix} = \gamma \Lambda \begin{pmatrix} \frac{\sigma_1^2}{\bar{x}_1^2 S_1^2} & \cdots & \frac{\rho_{1n}\sigma_1\sigma_n}{\bar{x}_1 S_1 \bar{x}_n S_n} \\ \vdots & & \vdots \\ \frac{\rho_{1n}\sigma_1\sigma_n}{\bar{x}_1 S_1 \bar{x}_n S_n} & \cdots & \frac{\sigma_n^2}{\bar{x}_n^2 S_n^2} \end{pmatrix} \left(\begin{pmatrix} \bar{x}_1 S_1 \\ \vdots \\ \bar{x}_n S_n \end{pmatrix} - W_0^M \frac{\lambda_M b}{a+b} \omega \right).$$

Simplifying further, we have

$$\begin{pmatrix} \mu_1 - \bar{x}_1 S_1 \\ \vdots \\ \mu_n - \bar{x}_n S_n \end{pmatrix} = \gamma \Lambda \begin{pmatrix} \frac{\sigma_1^2}{\bar{x}_1 S_1} & \cdots & \frac{\rho_{1n}\sigma_1\sigma_n}{\bar{x}_n S_n} \\ \vdots & & \vdots \\ \frac{\rho_{1n}\sigma_1\sigma_n}{\bar{x}_1 S_1} & \cdots & \frac{\sigma_n^2}{\bar{x}_n S_n} \end{pmatrix} \left(\begin{pmatrix} \bar{x}_1 S_1 \\ \vdots \\ \bar{x}_n S_n \end{pmatrix} - W_0^M \frac{\lambda_M b}{a+b} \omega \right),$$

which after plugging in

$$\omega = \frac{1}{\sum_{i=1}^n \mathbf{1}_i \bar{x}_i S_i} \begin{pmatrix} \mathbf{1}_1 \bar{x}_1 S_1 \\ \vdots \\ \mathbf{1}_n \bar{x}_n S_n \end{pmatrix}$$

gives us an implicit expression for share values:

$$\bar{x} \cdot S = \mu - \gamma \Lambda \Sigma \left(\mathbb{1} - \frac{\lambda_M b}{a+b} \frac{W_0^M}{\sum_i \mathbf{1}_i \bar{x}_i S_i} \mathbf{1_b} \right). \tag{B.1}$$

Notice that (B.1) is identical to our expression for share values (35) in the main model with $\mathbf{1_b} W_0^M / \sum_i \mathbf{1}_i \bar{x}_i S_i$ instead of $\mathbf{1_b}$.

Here, the value of assets under management, W_0^M, depends on asset prices. In general, (B.1) cannot be solved in closed form. Consider a special case when W_0^M consists only of the benchmark stocks, i.e., $W_0^M = \sum_i \mathbf{1}_i \bar{x}_i S_i$. Then (B.1) becomes exactly (35).

Lemmas 1 and 2 from the main text extend straightforwardly. The extensions of Lemma 3 and Propositions 1–3 are a bit more tricky in general, so we consider special cases.

First, start again with the case where $W_0^M = \sum_i \mathbf{1}_i \bar{x}_i S_i$. In this case, if firm i invests, then $\bar{x} \cdot S^{(i)}$ and $\bar{x}_i \Delta S_i$ are given exactly by (37) and (40), respectively. So Lemma 3 extends to this case. Performing the same analysis as in the main text, we get Proposition 1−3.

Somewhat more generally, suppose that $W_0^M = \xi \sum_i \mathbf{1}_i \bar{x}_i S_i + B_0$ so that the initial portfolio of the fund consists of the benchmark portfolio scaled by $\xi \geq 0$ and bond (or cash) holdings B_0. Assume for simplicity we assume that investment is financed by internal funds (or, equivalently, with the risk-free bond). Then the cost of investment to any firm is I (which is also true in our original model). We discuss at the end of this appendix what happens if investment is financed by equity instead.

Then equation (B.1) becomes

$$\bar{x} \cdot S = \mu - \gamma \Lambda \Sigma \left[\mathbb{1} - \frac{\lambda_M b}{a+b} \left(\xi + \frac{B_0}{\sum_i \mathbf{1}_i \bar{x}_i S_i} \right) \mathbf{1_b} \right].$$

Multiplying both sides by $\mathbf{1_b}^\top$ and denoting by $T = \sum_i \mathbf{1}_i \bar{x}_i S_i$ the total value of firms that are in the benchmark, we have that T is the positive root of the following quadratic equation:

$$T = \mu^\top \mathbf{1_b} - \gamma \Lambda \mathbf{1_b}^\top \Sigma \mathbb{1} + \gamma \Lambda \mathbf{1_b}^\top \Sigma \mathbf{1_b} \frac{\lambda_M b}{a+b} \left(\xi + \frac{B_0}{T} \right).$$

This delivers an explicit expression for asset prices:

$$\bar{x} \cdot S = \mu - \gamma \Lambda \Sigma \left[\mathbb{1} - \frac{\lambda_M b}{a+b} \left(\xi + \frac{B_0}{T} \right) \mathbf{1_b} \right].$$

If firm i invests,

$$\bar{x} \cdot S^{(i)} = \mu^{(i)} - I^{(i)} - \gamma \Lambda \Sigma^{(i)} \left[\mathbb{1} - \frac{\lambda_M b}{a+b} \left(\xi + \frac{B_0}{T^{(i)}} \right) \mathbf{1_b} \right].$$

where $T^{(i)} = \sum_j \mathbf{1}_j \bar{x}_j S_j^{(i)}$ is given by the positive root of

$$T^{(i)} = \left(\mu^{(i)} - I^{(i)} \right)^\top \mathbf{1_b} - \gamma \Lambda \mathbf{1_b}^\top \Sigma^{(i)} \mathbb{1} + \gamma \Lambda \mathbf{1_b}^\top \Sigma^{(i)} \mathbf{1_b} \frac{\lambda_M b}{a+b} \left(\xi + \frac{B_0}{T^{(i)}} \right). \tag{B.2}$$

The corresponding change in firm i's value is

$$\bar{x}_i \Delta S_i = \mu_y - I - \gamma\Lambda \sum_{j=1}^{n} \left[\rho_{jy}\sigma_j\sigma_y + (\sigma_y^2 + \rho_{iy}\sigma_i\sigma_y)\mathcal{I}_{j=i}\right] \left[1 - \frac{\lambda_M b}{a+b}\left(\xi + \frac{B_0}{T}\right)\mathbf{1}_j\right]$$
$$- \gamma\Lambda \sum_{j=1}^{n} \left[\rho_{jy}\sigma_j\sigma_y + (\sigma_y^2 + \rho_{iy}\sigma_i\sigma_y)\mathcal{I}_{j=i} + \rho_{ij}\sigma_i\sigma_j\right] \frac{\lambda_M b}{a+b} W_0^M \mathbf{1}_j \left[\frac{B_0}{T} - \frac{B_0}{T^{(i)}}\right],$$

where $\mathcal{I}_{j=i} = 1$ if $j = i$ and $\mathcal{I}_{j=i} = 0$ otherwise.

Suppose we have two firms i_{IN} and i_{OUT}, $i_{\text{IN}} \in \mathcal{B}$, $i_{\text{OUT}} \notin \mathcal{B}$ that are otherwise identical, i.e., $\sigma_{i_{\text{IN}}} = \sigma_{i_{\text{OUT}}} = \sigma$, $\rho_{i_{\text{IN}}y} = \rho_{i_{\text{OUT}}y} = \rho$ and $\rho_{i_{\text{IN}}j} = \rho_{i_{\text{OUT}}j} = \rho_j$ for $j \neq i_{\text{IN}}, i_{\text{OUT}}$. The expression for the benchmark inclusion subsidy is

$$\bar{x}_{i1}\Delta S_{i_{\text{IN}}} - \bar{x}_{i2}\Delta S_{i_{\text{OUT}}} = \left[\sigma_y^2 + \rho\sigma\sigma_y\right]\gamma\Lambda\frac{\lambda_M b}{a+b}\left(\xi + \frac{B_0}{T^{(i_{\text{IN}})}}\right)$$
$$- \left[\frac{B_0}{T^{(i_{\text{IN}})}} - \frac{B_0}{T^{(i_{\text{OUT}})}}\right]\gamma\Lambda\frac{\lambda_M b}{a+b}\sum_{j=1}^{n}\left(\rho_{jy}\sigma_j\sigma_y + \rho_j\sigma\sigma_j\right)\mathbf{1}_j.$$

The term in the first line is positive by Assumption 1. The term in the second line is new, and appears because the sum of benchmark weights is different depending on whether the investing firm is inside or outside the benchmark. It captures the fact that by investing, the firm grows and effectively reduces importance of other firms in the benchmark. It is natural to expect that this term in the second line is small. Formally, this term is proportional to $T^{(i_{\text{IN}})} - T^{(i_{\text{OUT}})} = o(T)$ when project y is small relative to T ($T^{(i_{\text{IN}})}$, $T^{(i_{\text{OUT}})}$, and T are all of the same order). So the term $1/T^{(i_{\text{IN}})} - 1/T^{(i_{\text{OUT}})}$ is $O(1/T^2)$. The rest of the second term, $\gamma\Lambda\lambda_M b/(a+b)B_0\sum_{j=1}^{n}\left(\rho_{jy}\sigma_j\sigma_y + \rho_j\sigma\sigma_j\right)\mathbf{1}_j$, is of the same order as $\bar{x}_{i_{\text{IN}}}S^{(i_{\text{IN}})}T$. So the second term is $O\left(\bar{x}_{i_{\text{IN}}}S^{(i_{\text{IN}})}/T\right)$, i.e., of the order of the benchmark weight $\omega_{i_{\text{IN}}}$.

The subsidy is still zero for risk-free projects. Indeed consider a special case when project y is risk free, i.e., $\sigma_y = 0$. It is easy to show that $T^{(i_{\text{OUT}})} = T$ for $i_{\text{OUT}} \in \{k+1, \ldots, n\}$. Moreover, suppose that $I = \mu_y$ so that there are no arbitrage opportunities. Then $\mu^{(i_{\text{IN}})} - I^{(i_{\text{IN}})} = \mu$ and $\Sigma^{(i_{\text{IN}})} = \Sigma$, and thus $T^{(i_{\text{IN}})} = T$ for $i_{\text{IN}} \in \{1, \ldots, k\}$. Hence for the risk-free project with $\mu_y = I$ we have $\Delta S_{i_{\text{IN}}} - \Delta S_{i_{\text{OUT}}} = 0$, i.e., both firms value it equally.

In our main model, whether investment is financed by debt or equity is irrelevant. In this specification, it is true if $W_0^M = \sum_i \mathbf{1}_i\bar{x}_i S_i$, but not in general. To see why, consider again the case with $W_0^M = \xi \sum_i \mathbf{1}_i \bar{x}_i S_i + B_0$ and suppose that investment I is financed by issuing $\delta_i = I/S_i^{(i)}$ additional shares. Then instead of $T^{(i)} = \sum_i \mathbf{1}_j x_j S_j^{(i)}$ we have $T^{(i)\prime} = \sum_{j \neq i} \mathbf{1}_j \bar{x}_j S_j^{(i)} + \mathbf{1}_i S_i^{(i)}(\bar{x}_i + \delta_i) = \sum_j \mathbf{1}_j \bar{x}_j S_j^{(i)} + \mathbf{1}_i I$, which is the positive root of

$$T^{(i)\prime} = \mu^{(i)\top}\mathbf{1_b} - \gamma\Lambda\mathbf{1}_\mathbf{b}^\top\Sigma^{(i)}\mathbb{1} + \gamma\Lambda\mathbf{1}_\mathbf{b}^\top\Sigma^{(i)}\mathbf{1_b}\frac{\lambda_M b}{a+b}\left(\xi + \frac{B_0}{T^{(i)\prime}}\right).$$

Comparing this equation to (B.2), one can see that $T^{(i)\prime} = T^{(i)}$ if i is outside the benchmark, and $T^{(i)\prime} > T^{(i)}$ if i is inside the benchmark. Using $T^{(i_{\text{OUT}})\prime} = T^{(i_{\text{OUT}})}$, the difference in the subsidies using debt vs. equity financing can be written as

$$\begin{aligned}
&(\bar{x}_{i1}\Delta S_{i_{\text{IN}}} - \bar{x}_{i2}\Delta S_{i_{\text{OUT}}})_{debt} - (\bar{x}_{i1}\Delta S_{i_{\text{IN}}} - \bar{x}_{i2}\Delta S_{i_{\text{OUT}}})_{equity} = \\
&= \gamma\Lambda\frac{\lambda_M b}{a+b}B_0\left[\frac{1}{T^{(i_{\text{IN}})}} - \frac{1}{T^{(i_{\text{IN}})\prime}}\right]\left\{\sigma_y^2 + \rho\sigma\sigma_y + \sum_{j=1}^n (\rho_{jy}\sigma_j\sigma_y + \rho_j\sigma\sigma_j)\,\mathbf{1}_j\right\}.
\end{aligned}$$

Using $T^{(i_{\text{IN}})\prime} > T^{(i_{\text{IN}})}$, the expression in the square brackets is strictly positive. Therefore the subsidy is larger with debt financing than with equity financing if and only if

$$B_0\left\{\sigma_y^2 + \rho\sigma\sigma_y + \sum_{j=1}^n (\rho_{jy}\sigma_j\sigma_y + \rho_j\sigma\sigma_j)\,\mathbf{1}_j\right\} > 0.$$

In particular, assuming that the expression in the curly brackets is strictly positive, the benchmark firm prefers financing investment with debt rather than equity if and only if the funds' initial portfolio includes positive bond holdings (in addition to the risky portfolio proportional to the benchmark portfolio). Notice that in the special case considered earlier in which $W_0^M = \sum_i \mathbf{1}_i \bar{x}_i S_i$ so that $B_0 = 0$, risk-free debt and equity financing are equivalent, i.e., deliver the same level of the benchmark inclusion subsidy. (The same result holds in our original model in the main text.) Empirically, B_0 is small, so we would expect the difference between risk-free debt vs. equity financing to be of second order.

In the most general case where W_0^M is the value of a general initial portfolio, the analysis is more complicated, but the overall message remains the same—whether risk-free debt or equity financing is cheaper depends on the composition of W_0^M.[1]

[1] We do not analyze risky debt, because the CARA framework with risky debt involves truncated normal

Appendix C

The goal of this appendix is to discuss how one would approach investigating the welfare implications of the benchmark inclusion subsidy. We will argue that within the current model, such implications are ambiguous even in the simplest case. To do a proper analysis, one would need to endogenize optimal contracts between fund investors and their managers.

To illustrate the ambiguity of welfare effects of the benchmark inclusion subsidy, consider a simple version of our model, where the setup is similar to the one in Section 3, but there are no direct investors—only fund investors and fund managers (direct investors can be easily incorporated, as discussed at the end of this appendix). We will consider potential mergers of firms 1 and 2 with firm y. In this case, total cash flows in the economy would not change after a merger, and neither will the total holdings of the fund, since there are no direct investors and the fund has to hold the total supply of shares. Therefore, the only difference in the social welfare (which we will measure as sum of utilities of the fund manager and fund investor) might come from a change in the risk sharing between these two parties. We will show that when a non-benchmark firm 2 acquires y, there is no change in the utilities of each agent and hence no change in the social welfare. However, when the benchmark firm 1 makes the acquisition, part of the risk will be moved from the manager (who wants to hold less of the more expensive asset) to the fund investor.

Let us introduce some notation. Denote by $x^M_{-1,i}$ and $x^F_{-1,i}$ the initial endowments of risky asset i by managers and fund investors, respectively. Also, let $z^M_1 = (a+b)x^M_1 - b$, $z^M_i = (a+b)x^M_i$, $i = 2, y$, denote the effective asset holdings of the manager. The corresponding effective asset holdings of the fund investors are $z^F_i = x^M_i - z^M_i$, $i = 1, 2, y$, where x^M_i is the manager's demand for asset i given by equation (2) in the paper.

Prior to a merger, the utilities of managers and fund investors (in the mean-variance form) are

$$U^M = \sum_{i=1,2,y} \left[\left(x^M_{-1,i} - z^M_i \right) S_i + z^M_i \mu_i - \frac{\gamma}{2} \left(z^M_i \right)^2 \sigma^2_i \right] + c,$$

distributions, which makes the analysis intractable.

$$U^F = \sum_{i=1,2,y} \left[\left(x^F_{-1,i} - z^F_i \right) S_i + z^F_i \mu_i - \frac{\gamma}{2} \left(z^F_i \right)^2 \sigma^2_i \right] - c.$$

We construct a social welfare function which applies equal weights to the (mean-variance) utilities of different agents:

$$U^M + U^F = \sum_{i=1,2,y} \left\{ \mu_i - \frac{\gamma}{2} \left[\left(z^M_i \right)^2 + \left(z^F_i \right)^2 \right] \sigma^2_i \right\}.$$

Notice that such choice of weights make the terms $\left(x^M_{-1,i} - z^M_i \right) S_i$ and $\left(x^F_{-1,i} - z^F_i \right) S_i$ wash out of the social welfare function (due to market clearing), and the terms $z^M_i \mu_i$ and $z^F_i \mu_i$ sum up to a constant. These terms capture simple redistribution of (expected) resources across the agents, so they represent movement "along the frontier," while changes to $U^M + U^F$ will represent movement inside or outside the frontier. The direction of this redistribution effect depends (in part) on the agents' initial endowments. If an agent is endowed with a large (small) amount of the benchmark firm's stock, s/he is going to benefit (lose) from the subsidized increase in the price of this stock following an investment or merger.

As for the (aggregate) effect of the subsidy on the social welfare, in this example it only comes from the shift of risk from one group of agents to the other, as captured by the second term in the above expression for $U^M + U^F$. In what follows, we will explore how this term changes depending on which firm acquires y.

It will be useful to note that in this simple example without direct investors (so that $\lambda_M = \lambda_F = 1/2$), the equilibrium allocations are very simple, and are given by $x^M_i = 2$, $i = 1, 2, y$, and $z^M_1 = 2(a + b) - b$, $z^M_2 = z^M_y = 2(a + b)$, $z^F_1 = 2(1 - a - b) + b$, $z^M_2 = z^M_y = 2(1 - a - b)$.

We will now consider the effects of a merger on the social welfare. Suppose first that firm 2 (outside the benchmark) acquires firm y. It is easy to verify that the equilibrium holding of assets (1 and 2, since asset y has been acquired) do not change, i.e., $v'_i = v_i$ for $v \in \{x^M, z^M, z^F\}$, $i = 1, 2$, in particular,

$$\begin{aligned} z'^M_1 &= 2(a + b) - b, \quad z'^M_2 = 2(a + b), \\ z'^F_1 &= 2(1 - a - b) + b, \quad z'^M_2 = 2(1 - a - b). \end{aligned} \tag{C.1}$$

It is easy to see that in this case, the utilities of the manager and fund investor after the merger are the same as before the merger. The sum of the two utilities is also the same as before, and is equal to

$$\begin{aligned}\left(U^M + U^F\right)_2' = U^M + U^F = \mu_1 + \mu_2 + \mu_y - \frac{\gamma}{2}\left\{[2(a+b)-b]^2 + [2(1-a-b)+b]^2\right\}\sigma_1^2 \\ - \frac{\gamma}{2}\left\{[2(a+b)]^2 + [2(1-a-b)]^2\right\}\left(\sigma_2^2 + \sigma_y^2\right).\end{aligned}$$

Now suppose that firm 1 (inside the benchmark) acquires y. In this case, (C.1) above still holds, but the sum of utilities changes to

$$\begin{aligned}\left(U^M + U^F\right)_1' = \mu_1 + \mu_2 + \mu_y - \frac{\gamma}{2}\left\{[2(a+b)-b]^2 + [2(1-a-b)+b]^2\right\}\left(\sigma_1^2 + \sigma_y^2\right) \\ - \frac{\gamma}{2}\left\{[2(a+b)]^2 + [2(1-a-b)]^2\right\}\sigma_2^2.\end{aligned}$$

What changes is how the risk associated with y's cashflows gets allocated between the manager and the fund investor. Before the merger, the corresponding utility terms for the manager and investor are proportional to $-[2(a+b)]^2$ and $-[2(1-a-b)]^2$, respectively. After the acquisition by firm 1, those terms change to $-[2(a+b)-b]^2$ and $-[2(1-a-b)+b]^2$, so part of the risk is shifted from the manager to the fund investor.

To see if the social welfare increases or decreases as a result, compute

$$\begin{aligned}&\left(U^M + U^F\right)_1' - \left(U^M + U^F\right) = \\ &= \frac{\gamma\sigma_y^2}{2}\left\{[2(a+b)]^2 + [2(1-a-b)]^2 - [2(a+b)-b]^2 - [2(1-a-b)+b]^2\right\} \\ &= \gamma\sigma_y^2 b\left\{4\left[2(a+b)-1\right] - b\right\}.\end{aligned}$$

The sign of the above expression is uncertain, and depends on the parameters of the compensation contract.[2] To assess the effect of the benchmark inclusion subsidy on welfare even in this simplest case, we would need to model how the fund investors optimally design compensation contracts for the fund managers. Intuitively, to provide incentives to the

[2] Kashyap et al. (2020) show that in the optimal contract, $a+b > 1/2$ and $b > 0$. Notice, however, that this is not enough to sign the above expression.

manager (given an incentive problem that we do not model here), risk sharing between her and the fund investor would be distorted away from perfect risk sharing. Whether the shift of risk from the manager to fund investor due to acquisition helps or aggravates the incentive problem in unclear without explicitly modeling it.[3]

Next, suppose we add direct investors to the above analysis. The main difference is that the fund's risky asset holdings are no longer equal to the fixed net supply of the stock. However, just as in the previous case, it is easy to show that when the non-benchmark firm 2 acquires y, the equilibrium asset holdings of the agents remain the same as before the merger, and thus so do their utilities.

Suppose instead that the benchmark firm 1 acquires y. The stock price of firm 2 does not change, and neither do the agents' equilibrium holdings of asset 2. The price of asset-1 goes up (above the sum of S_1 and S_y, due to the benchmark inclusion subsidy), and in equilibrium, the asset 1 holdings of direct investors, x_1^D (because of the price effect), decrease, while the total fund asset 1 holdings, $x_1^M = z_1^M + z_1^F$, increase (because of the inelastic demand component). Thus we have the shift of risk from the direct investors to the fund as a whole. Moreover, we show in our main analysis (see the proof of Lemma 1) that $x_i^D = z_i^M$ for all i, and thus the risk component of the manager's utility always coincides with that of the direct investor, while the extra risk borne by the fund is carried by the fund investor. So, after the merger of firm 1 and y, the risk is shifted from the direct investor and fund manager to the fund investor. (Notice that it does not mean that the fund investor is worse off, as his utility also has the expected component part, which washes out from the social welfare function through the redistribution effect.) As before, it is hard to sign the total effect on social welfare without knowing more about the contract.

We have illustrated ambiguity of welfare implications of the benchmark inclusion subsidy in a simple setting with mergers, where the total cash flows remain unchanged after a merger. The analysis with investments into new projects would have an additional layer of complexity.

[3]Kashyap et al. (2020) analyze the optimal contract design in a similar setting, but do not consider corporate decisions. That environment is much more complicated than what we study here. So properly doing welfare analysis would require imposing a lot more additional structure.

Appendix D

In this appendix we analyze the benchmark inclusion subsidy in a setting in which benchmarking affects second moments of equity returns.

Consider a generalization of our model to three periods, $t = 0, 1, 2$. Direct investors derive utility from terminal wealth at the end of period 2. Fund managers derive utility from the compensation, which is paid at the end of period 2. Both have CARA preferences with the risk aversion coefficient γ.

There is a riskless asset (paying an interest rate that is normalized to zero), and n risky assets. Risky assets pay dividends at the end of period 2, but there is cash-flow news that arrives in period 1. This news and the terminal dividends are given by $D_{i,t} = c_i z_t + \varepsilon_{i,t}, \ \ i = 1, \ldots, n, \ \ t = 1, 2$, where z_t is aggregate shock, $\varepsilon_{i,t}$ is an idiosyncratic shock and c_i is the loading on the aggregate shock. We follow Buffa et al. (2014) and assume that the variables z_t and ε_{it} follow square-root processes:[4]

$$
\begin{aligned}
&z_{t+1} = \mu_z z_t + \sigma_z \sqrt{z_t}\eta_{t+1}, \ \eta_t \sim N(0,1), \ \ t = 0,1, \\
&\varepsilon_{i,t+1} = \mu_{\varepsilon,i}\varepsilon_{i,t} + \sigma_{\varepsilon,i}\sqrt{\varepsilon_{i,t}}\eta_{i,t+1}, \eta_{i,t} \sim N(0,1), \ t = 0,1, \ i = 1,...,n,
\end{aligned}
$$

where $\eta_{i,t+1}$ are i.i.d. and independent of the aggregate shock η_t, and μ_z, σ_z, z_0, $\mu_{\varepsilon,i}$, $\sigma_{\varepsilon,i}$, $\varepsilon_{i,0}$, $i = 1,...,n$, are positive scalars. At $t = 2$, the risky stocks pay off terminal dividends given by $D_2 = (D_{1,2}, \ldots, D_{n,2})$, which is an $n \times 1$ vector. With this specification, $D_{i,t+1} \sim N\left(c_i\mu_z z_t + \mu_{\varepsilon,i}\varepsilon_{i,t}, \, c_i^2\sigma_z^2 z_t + \sigma_{\varepsilon,i}^2\varepsilon_{i,t}\right)$ and $Cov_t(D_{i,t+1}, D_{j,t+1}) = c_i c_j \sigma_z^2 z_t$, $t = 0,1$, $i = 1, \ldots, n$.

It is convenient to introduce the notation $\Sigma_t \equiv \Sigma_z z_t + \Sigma_{\varepsilon,t}$ for the variance-covariance matrix of D_{t+1}, conditional on information available at time t, where the $n \times n$ matrix Σ_z has the (i,j)-th element equal to $c_i c_j \sigma_z^2$, and the $n \times n$ matrix $\Sigma_{\varepsilon t}$ is a diagonal matrix that has the (i,i)-th element equal to $\sigma_{\varepsilon,i}^2\varepsilon_{i,t}$. Also, denote by c and ε_t the $n \times 1$ vectors of c_i and $\varepsilon_{i,t}$, $i = 1, \ldots, n$, respectively.

[4]The usual caveat about the quantity inside the square root potentially becoming negative applies. See Backus et al. (2001), who use a similar square-root process in discrete time, for a discussion. The problem goes away in a continuous-time version of this model, as in Buffa et al., which is tractable, but takes us too far away from our baseline setting.

The asset prices in period 1 are given by

$$S_1 = \mu_z c\, z_1 + \mu_\varepsilon \epsilon_1 - \gamma\Lambda\left(\Sigma_z z_1 + \Sigma_{\varepsilon,1}\right)\left(\mathbb{1} - \frac{\lambda_M b}{a+b}\mathbf{1_b}\right).$$

The difference in return volatilities for firms inside and outside the benchmark comes from the term highlighted in red.

The asset price of asset i in period zero is given by

$$\begin{aligned} S_{i,0} = &\left[\mu_z c_i - \gamma c_i c^\top\left(\mathbb{1} - \frac{\lambda_M b \mathbf{1_b}}{a+b}\right)\Lambda\sigma_z^2\right]\left\{\mu_z - \gamma\left[\left(\mathbb{1} - \frac{\lambda_M b\mathbf{1_b}}{a+b}\right)^\top \Lambda\mu_z c - B_z\right]\sigma_z^2\right\} z_0 \\ &+ \left[\mu_{\varepsilon,i} - \gamma\left(1 - \frac{\lambda_M b\mathbf{1_b}_i}{a+b}\right)\Lambda\sigma_{\varepsilon,i}^2\right]\left\{\mu_{\varepsilon,i} - \gamma\left[\left(1 - \frac{\lambda_M b\mathbf{1_b}_i}{a+b}\right)\Lambda\mu_{\varepsilon,i} - B_{\varepsilon,i}\right]\sigma_{\varepsilon,i}^2\right\}\varepsilon_{i,0}, \end{aligned}$$

where

$$\begin{aligned} B_z &= \frac{\gamma}{2}\left(\mathbb{1} - \lambda_M\frac{b\mathbf{1_b}}{a+b}\right)^\top \Sigma_z\left(\mathbb{1} - \lambda_M\frac{b\mathbf{1_b}}{a+b}\right)\Lambda^2, \\ B_{\varepsilon,i} &= \frac{\gamma}{2}\left(1 - \lambda_M\frac{b\mathbf{1_b}_i}{a+b}\right)^2 \sigma_{\varepsilon,i}^2\Lambda^2, \end{aligned}$$

and $\mathbf{1_b}_i$ is the i-th component of vector $\mathbf{1_b} = (\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{n-k})$. As before, the red terms trace the effects of the difference in return volatility of firms inside and outside the benchmark.

It is straightforward to show that the variances and the absolute value of covariances of per-share returns, $Var(S_{j,1} - S_{j,0})$ and $|Cov(S_{i,1} - S_{i,0}, S_{j,1} - S_{j,0})|$, $i, j = 1, \dots, n$, are higher for stocks in the benchmark.

Next, we will investigate the valuation of an investment project by different firms. Consider a project y that requires initial investment I. We assume that if firm j adopts the project, then firm j's cash flows become

$$\begin{aligned} D_{j,t}^{(j)} &= (c_j + c_y)z_t + \varepsilon_{j,t}^{(j)}, \\ \varepsilon_{j,t+1}^{(j)} &= (\mu_{\varepsilon,j} + \mu_{\varepsilon,y})\hat{\varepsilon}_{j,t} + (\sigma_{\varepsilon,j} + \sigma_{\varepsilon,y})\sqrt{\varepsilon_{j,t}^{(j)}}\eta_{j,t+1}. \end{aligned}$$

Denote $c_i^{(j)} = c_i \mathbb{1}_{i \neq j} + (c_i + c_y)\mathbb{1}_{i=j}$, $\mu_{\varepsilon,i}^{(j)} = \mu_{\varepsilon,i}\mathbb{1}_{i \neq j} + (\mu_{\varepsilon,i} + \mu_{\varepsilon,y})\mathbb{1}_{i=j}$, $\sigma_{\varepsilon,i}^{(j)} = \sigma_{\varepsilon,i}\mathbb{1}_{i \neq j} + (\sigma_{\varepsilon,i} + \sigma_{\varepsilon,y})\mathbb{1}_{i=j}$, $\varepsilon_{i,t}^{(j)} = \varepsilon_{i,t}\mathbb{1}_{i \neq j} + \varepsilon_{j,t}^{(j)}\mathbb{1}_{i=j}$, where $\mathbb{1}$ is the indicator function. Also let $\Sigma_t^{(j)} \equiv \Sigma_z^{(j)} z_t + \Sigma_{\varepsilon,t}^{(j)}$, where $\Sigma_z^{(j)}$ is an $n \times n$ matrix with the (i,k)-th element equal to $c_i^{(j)} c_k^{(j)} \sigma_z^2$, and $\Sigma_{\varepsilon,t}^{(j)}$ is the $n \times n$ matrix diagonal matrix that has the (i,i)-th element equal to $\left(\sigma_{\varepsilon,i}^{(j)}\right)^2 \varepsilon_{i,t}^{(j)}$. Also, denote by $c^{(j)}$ the $n \times 1$ vector of $c_i^{(j)}$. Finally, denote

$$
\begin{aligned}
B_z^{(j)} &= \frac{\gamma}{2}\left(\mathbb{1} - \lambda_M \frac{b\mathbf{1_b}}{a+b}\right)^\top \Sigma_z^{(j)} \left(\mathbb{1} - \lambda_M \frac{b\mathbf{1_b}}{a+b}\right)\Lambda^2, \\
B_{\varepsilon,i}^{(j)} &= \frac{\gamma}{2}\left(1 - \lambda_M \frac{b\mathbf{1_b}_i}{a+b}\right)^2 \left(\sigma_{\varepsilon,i}^{(j)}\right)^2 \Lambda^2,
\end{aligned}
$$

Using these notations, the change in the stockholder value, $\Delta S_{j,0} = S_{j,0}^{(j)} - S_{j,0}$, is

$$
\begin{aligned}
\Delta S_{j,0} = {} & -I + \left[\mu_z c_y - \gamma c_y c^\top \left(1 - \frac{\lambda_M b\mathbf{1_b}}{a+b}\right)\Lambda\sigma_z^2 - \gamma c_y c_j \left(1 - \frac{\lambda_M b\mathbf{1_b}_j}{a+b}\right)\Lambda\sigma_z^2\right] \times \\
& \times \left\{\mu_z - \gamma\left[\left(\mathbb{1} - \frac{\lambda_M b\mathbf{1_b}}{a+b}\right)^\top \Lambda\mu_z c^{(j)} - B_z^{(j)}\right]\sigma_z^2\right\} z_0 \\
& + \left[\mu_z c_j - \gamma c_j c^\top \left(1 - \frac{\lambda_M b\mathbf{1_b}}{a+b}\right)\Lambda\sigma_z^2\right]\gamma\left[\left(\mathbb{1} - \frac{\lambda_M b\mathbf{1_b}}{a+b}\right)^\top \Lambda\mu_z\left(c - c^{(j)}\right) - B_z + B_z^{(j)}\right]\sigma_z^2 z_0 \\
& + \left[\mu_{\varepsilon,y} - \gamma\left(1 - \frac{\lambda_M b\mathbf{1_b}_j}{a+b}\right)\Lambda\left(\sigma_{\varepsilon,y}^2 + 2\sigma_{\varepsilon,j}\sigma_{\varepsilon,y}\right)\right] \times \\
& \times \left\{\mu_{\varepsilon,j} + \mu_{\varepsilon,y} - \gamma\left[\left(1 - \frac{\lambda_M b\mathbf{1_b}_j}{a+b}\right)\Lambda(\mu_{\varepsilon,j} + \mu_{\varepsilon,y}) - B_{\varepsilon,j}^{(j)}\right](\sigma_{\varepsilon,j} + \sigma_{\varepsilon,y})^2\right\}\varepsilon_{j,0} \\
& + \left[\mu_{\varepsilon,j} - \gamma\left(1 - \frac{\lambda_M b\mathbf{1_b}_j}{a+b}\right)\Lambda\sigma_{\varepsilon,j}^2\right] \times \\
& \times \left\{\mu_{\varepsilon,y} - \gamma\left[\left(1 - \frac{\lambda_M b\mathbf{1_b}_j}{a+b}\right)\Lambda(\mu_{\varepsilon,j} + \mu_{\varepsilon,y}) - B_{\varepsilon,j}^{(j)}\right](\sigma_{\varepsilon,j} + \sigma_{\varepsilon,y})^2 \right. \\
& \quad \left. + \gamma\left[\left(1 - \frac{\lambda_M b\mathbf{1_b}_j}{a+b}\right)\Lambda\mu_{\varepsilon,j} - B_{\varepsilon,j}\right]\sigma_{\varepsilon,j}^2\right\}\varepsilon_{j,0}.
\end{aligned}
$$

Consider two otherwise identical firms, one of which is in the benchmark and the one is outside. The benchmark inclusion subsidy is given by

$$
\begin{aligned}
&\Delta S_{i_{IN},0} - \Delta S_{i_{OUT},0} = \\
&= \gamma c_y c_{i_{IN}} \frac{\lambda_M b}{a+b} \Lambda \sigma_z^2 \left\{ \mu_z - \gamma \left[\left(\mathbb{1} - \frac{\lambda_M b \mathbf{1_b}}{a+b} \right)^\top \Lambda \mu_z c^{i(IN)} - B_z^{(i_{IN})} \right] \sigma_z^2 \right\} z_0 \\
&+ \left[\mu_z c_y - \gamma c_y c^\top \left(1 - \frac{\lambda_M b \mathbf{1_b}}{a+b} \right) \Lambda \sigma_z^2 - \gamma c_y c_{i_{IN}} \Lambda \sigma_z^2 \right] \times \\
&\quad \times \gamma \left[\frac{\lambda_M b}{a+b} \Lambda \mu_z c_y - \frac{\gamma}{2} \Lambda^2 c_y^2 \left(2 - \lambda_M \frac{b}{a+b} \right) \frac{\lambda_M b}{a+b} - \gamma \Lambda^2 c_y \frac{\lambda_M b}{a+b} c^\top \left(\mathbb{1} - \frac{\lambda_M b \mathbf{1_b}}{a+b} \right) \right] \sigma_z^2 z_0 \\
&+ \left[\mu_z c_{i_{IN}} - \gamma c_{i_{IN}} c^\top \left(1 - \frac{\lambda_M b \mathbf{1_b}}{a+b} \right) \Lambda \sigma_z^2 \right] \times \\
&\quad \times \gamma \left[\frac{\lambda_M b}{a+b} c_y \Lambda \mu_z - \frac{\gamma}{2} \Lambda^2 c_y^2 \left(2 - \frac{\lambda_M b}{a+b} \right) \frac{\lambda_M b}{a+b} - \gamma \Lambda^2 c_y \frac{\lambda_M b}{a+b} c^\top \left(\mathbb{1} - \frac{\lambda_M b \mathbf{1_b}}{a+b} \right) \right] \sigma_z^2 z_0 \\
&+ \gamma \left(\sigma_{\varepsilon,y}^2 + 2\sigma_{\varepsilon,i_{IN}} \sigma_{\varepsilon,y} \right) \frac{\lambda_M b}{a+b} \Lambda \times \\
&\quad \times \left\{ \mu_{\varepsilon,i_{IN}} + \mu_{\varepsilon,y} - \gamma \left[\left(1 - \frac{\lambda_M b}{a+b} \right) \Lambda (\mu_{\varepsilon,i_{IN}} + \mu_{\varepsilon,y}) - B_{\varepsilon,i_{IN}}^{(i_{IN})} \right] (\sigma_{\varepsilon,i_{IN}} + \sigma_{\varepsilon,y})^2 \right\} \varepsilon_{i_{IN},0} \\
&+ \left(\mu_{\varepsilon,y} - \gamma (\sigma_{\varepsilon,y}^2 + 2\sigma_{\varepsilon,i_{IN}} \sigma_{\varepsilon,y}) \Lambda \right) \times \\
&\quad \times \gamma \left[\frac{\lambda_M b}{a+b} \Lambda (\mu_{\varepsilon,i_{IN}} + \mu_{\varepsilon,y}) - \frac{\gamma}{2} \frac{\lambda_M b}{a+b} \left(2 - \frac{\lambda_M b}{a+b} \right) (\sigma_{\varepsilon,i_{IN}} + \sigma_{\varepsilon,y})^2 \right] (\sigma_{\varepsilon,i_{IN}} + \sigma_{\varepsilon,y})^2 \varepsilon_{i_{IN},0} \\
&+ \gamma \sigma_{\varepsilon,i_{IN}}^2 \frac{\lambda_M b}{a+b} \Lambda \left\{ \mu_{\varepsilon,y} - \gamma \left[\left(1 - \frac{\lambda_M b}{a+b} \right) \Lambda (\mu_{\varepsilon,i_{IN}} + \mu_{\varepsilon,y}) - B_{\varepsilon,i_{IN}}^{(i_{IN})} \right] (\sigma_{\varepsilon,i_{IN}} + \sigma_{\varepsilon,y})^2 \right\} \varepsilon_{i_{IN},0} \\
&+ \gamma \sigma_{\varepsilon,i_{IN}}^2 \frac{\lambda_M b}{a+b} \Lambda \gamma \left(\left(1 - \frac{\lambda_M b}{a+b} \right) \Lambda \mu_{\varepsilon,i_{IN}} - B_{\varepsilon,i_{IN}} \right) \sigma_{\varepsilon,j}^2 \varepsilon_{j,0} \\
&+ (\mu_{\varepsilon,i_{IN}} - \gamma \sigma_{\varepsilon,i_{IN}}^2 \Lambda) \times \\
&\quad \times \left\{ \frac{\lambda_M b}{a+b} \Lambda \mu_{\varepsilon,y} - \frac{\gamma}{2} \frac{\lambda_M b}{a+b} \left(2 - \frac{\lambda_M b}{a+b} \right) \left[(\sigma_{\varepsilon,i_{IN}} + \sigma_{\varepsilon,y})^2 + \sigma_{\varepsilon,y}^2 \right] \right\} \left(2\sigma_{\varepsilon,i_{IN}} \sigma_{\varepsilon,y} + \sigma_{\varepsilon,y}^2 \right) \varepsilon_{i_{IN},0}.
\end{aligned}
$$

Again, the red terms are parts of the subsidy coming from the fact that firms inside and outside the benchmark have different return volatility. They are derived from the terms $B_z^{(i_{IN})} - B_z^{(i_{OUT})}$, $B_{\varepsilon,i_{IN}} - B_{\varepsilon,i_{OUT}}$ and $B_{\varepsilon,i_{IN}}^{(i_{IN})} - B_{\varepsilon,i_{OUT}}^{(i_{OUT})}$. When adopting a new project, a firm inside the benchmark would face a larger increase in return volatility than a firm outside the benchmark; this reduces its period-0 price and reduces the subsidy size. Hence this effect of return volatility lowers the benchmark subsidy.[5]

[5]Note that empirically we typically observe a *decrease* in volatility in response to index/benchmark inclusion (e.g., for the S&P 500 additions). This is because index inclusion often coincides with an improvement in a stock's liquidity, which lowers the volatility of the stock's returns. This effect works in the

However, there are other effects of volatility. In particular, the period-0 return volatility also affects the *level* of prices and hence the subsidy. For example, as in the main model, part of the subsidy comes from the fact that benchmark firms are penalized less for the cash-flow variance. The return volatility impacts the strength of this channel too. The corresponding terms that are highlighted in blue. These terms would be present even if the two firms, one inside and the other outside the benchmark, had the same period-zero return volatility. The overall effect of return volatility, is, therefore, too complicated to sign.

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opposite direction to the one we identified above. We abstract away from liquidity considerations in our model, and hence our analysis of the second moment effects on the subsidy is subject to that important caveat. One paper that disentangles the liquidity and index membership effects is Ben-David et al. (2018), who document that ETF membership increases stock return volatility.